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A simple quantum picture for the Petermann excess noise factor

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Abstract. We present a very simple "toy model" that allows one to recover the essential mathematical and physical features that lead to excess quantum noise and excess linewidth in lasers with unstable resonators. In a consistent quantum mechanical description, the excess noise factor (Petermann factor) is physically attributed to loss-induced coupling between the cavity eigenmodes.

PACS. 42.40.Lc Quantum fluctuations, quantum noise, and quantum jumps – 42.55.Ah General laser theory – 42.60.Da Resonators, cavities, amplifiers, arrays, and rings

1 Introduction

There has been recently a widespread interest in the theoretical and experimental studies of the excess quantum noise which appears in lasers with unstable cavities [1–3]. One striking feature of such systems is that the laser linewidth can be much larger than the usual Schawlow-Townes linewidth, by a factor which is known as the Petermann excess noise factor K [4]. In recent experiments with suitably designed unstable laser cavities [1,2], K can be as large as several hundreds; it is therefore a very large unambiguous effect, which deserves thorough attention.

From a theoretical point of view, the essential feature of unstable resonators, which is intimately related with the existence of the Petermann factor, is their non-hermitian character [3]: losses, due to the aperturing within the cavity, play an essential role in the definition of the laser modes (see Fig. 1). This feature has very important consequences. First, the relation between input and output modes in a cavity roundtrip is non-unitary, and it is also non invariant under propagation reversal (see Fig. 1). Second, as it was analyzed in detail by Siegman [3], one can define laser (transverse) eigenmodes as modes which reproduce in shape after one roundtrip in the cavity, up to a complex multiplicative constant. Then the set of laser eigenmodes $\{\Psi_n\}$ is non-orthogonal, but it is bi-orthogonal to another "adjoint" set of modes $\{\Phi_n\}$, where the set $\{\Phi_n^*\}$ is obtained by reverting the direction of propagation in the cavity. One has therefore:

$$(\Psi_n, \Psi_n) = 1 \qquad (\Psi_n, \Phi_m) = \delta_{nm} \tag{1}$$

where (,) denotes the hermitian scalar product. It follows from equation (1) that the modes of the set $\{\Phi_n\}$ cannot be normalized. For a given resonator, the set of modes $\{\Psi_n\}$ and $\{\Phi_n\}$ can be calculated numerically; it is then straightforward to calculate K, which is simply given for mode n by [1,5]:

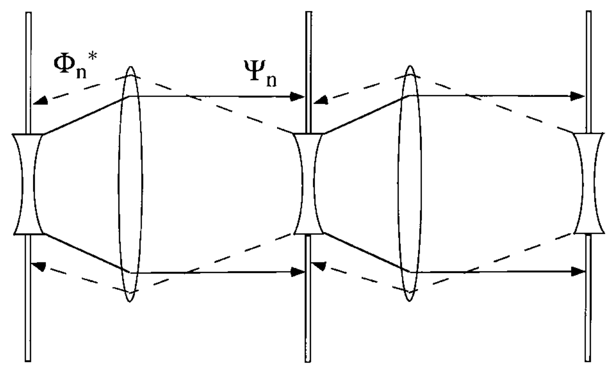

Fig. 1. Unfolded beam paths in an unstable resonator (see [3]). The aperturing losses are important at each roundtrip, and the forward and backward modes have different shapes

$$K_n = (\Phi_n, \Phi_n). \tag{2}$$

Though this picture works quite efficiently, and is in very good agreement with the experiments, it has some built-in conceptual difficulties. The main one is how to turn this semi-classical model into a fully quantum description: the complex amplitudes of a set of classical non-orthogonal modes cannot be turned into a set of non-commuting operators [6], because of problems related to unitarity (such a procedure would violate the conservation of probability). As a possible solution to this difficulty, it is well known in quantum optics that losses and gain can be conveniently described by introducing appropriate "vacuum modes" [7], that allow one to recover the unitarity of the input-output scattering matrix [8,9]. Such an approach is closely related to the so-called "linear input-output method" [10,11].

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The purpose of the present paper is to describe a very simple "toy model", where this procedure can be explicitly carried out. The non-unitary scattering matrix in a (truncated) set of modes then appears again to be unitary when the set is extended to include modes in loss and gain channels. The Petermann factor is then calculated both by the standard way (in the truncated set) and by a fully quantum way (in the complete set). Both results will be shown to be the same, and a clear physical explanation will come out for the Petermann excess noise factor: it can actually be attributed to a coupling between the "laser" modes (modes of the truncated set), which occurs in the presence of the "loss" modes (modes of the complete set which are not included in the truncated set). The resulting notion of "loss-induced coupling" will be precised mathematically later on.

The paper is organized as follow: in Section 2, we describe the model, which exhibits non-orthogonal modes in a semi-classical approach, and we calculate the excess noise factor by using equation (2). In Section 3, we present the quantum approach for the same model, and we recover the excess noise factor by two approaches: either from a direct quantum calculation, or by re-demonstrating directly equation (2) in a quantum framework. The relevance of the model to actual laser resonators is discussed in Section 4, and some technical details are given in Appendix A and B.

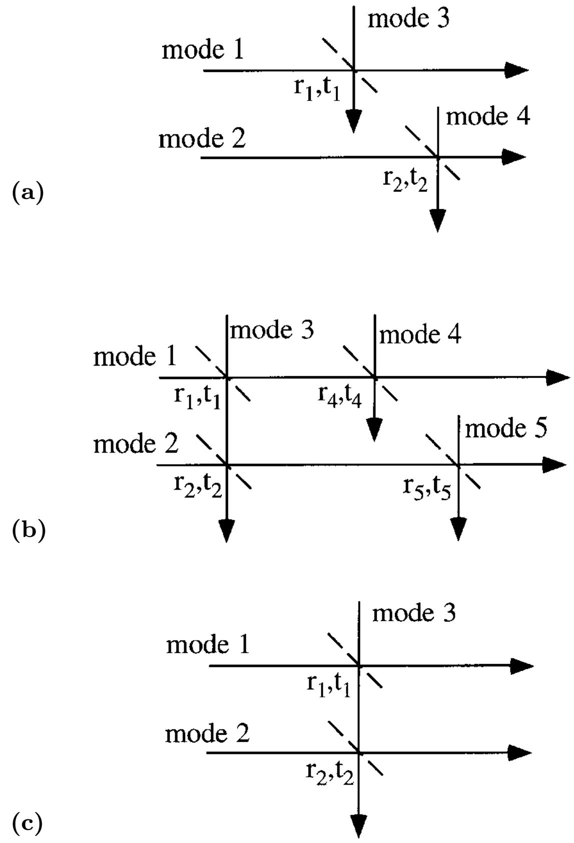

Fig. 2. (a): Two "laser" modes (horizontal arrows) with two "loss" modes (vertical arrows). (b): Two "laser" modes (horizontal arrows) with three "loss" modes (vertical arrows). Mode 3 couples the two laser modes 1 and 2 (loss-induced coupling). (c): Simplified case: two "laser" modes (horizontal arrows) coupled through one loss mode.

2 Non-orthogonality *via* truncated scattering matrices

In order to build our "toy model", we will use a recipe well known in quantum optics: an optical loss can be represented by a beamsplitter which takes out a part of the beam. Since the amplitude of the beam is reduced, unitarity is preserved by introducing a "vacuum" mode which enters from the other channel of the beamsplitter [8]. This procedure can be seen as an "automatic" way to fulfill the fluctuation-dissipation theorem in the scattering process, and therefore to ensure the conservation of probability. For the sake of clarity, we point out that the losses considered in this section are intracavity losses from the point of view of the laser, that is, light which is lost somewhere in the cavity and is not part of the output beam. The treatment of the laser output coupler will be carried out later on. In order to make the model as simple as possible, we will consider only two "laser" modes, which can be seen for example as two different spatial modes. Both modes have losses, modelled by beamsplitters as said previously. The simplest scheme is then the one shown in Figure 2a, with two "laser" modes and two loss modes. If this four-mode system is truncated to two, it becomes non-unitary, but nothing special happens: the two laser modes remain orthogonal, and evolve independantly with their own losses. A much more interesting configuration is the one presented in Figure 2b, with five modes: it amounts to say that some of the light which is taken out from mode 1 is recoupled into mode 2. Using the mirrors transmission and reflection coefficients shown in Figure 2b (t_i, r_i with $t_i^2 + r_i^2 = 1$ and $i = 1, 5$), the five-mode unitary scattering matrix can be written:

$$\mathbf{M_5} = \begin{pmatrix} t_1 t_4 & 0 & r_1 t_4 & r_4 & 0 \\ -r_1 r_2 t_5 & t_2 t_5 & t_1 r_2 t_5 & 0 & r_5 \\ -r_1 t_2 & -r_2 & t_1 t_2 & 0 & 0 \\ -t_1 r_4 & 0 & -r_1 r_4 & t_4 & 0 \\ r_1 r_2 r_5 & -t_2 r_5 & -t_1 r_2 r_5 & 0 & t_5 \end{pmatrix}. \qquad (3)$$

In the standard semiclassical description [1], this scattering matrix truncated to modes 1 and 2 describes a roundtrip inside the laser cavity. The "laser" modes are then the eigenvectors of the upper left 2×2 matrix extracted from the above one. This truncated matrix is not unitary, and its eigenvectors are generally non-orthogonal. Conspicuous features of non-hermitian resonators then obviously show up. In the following, we will simplify even further the system, in order to carry out explicitly all calculations, while keeping the interesting part of the physics. We shall therefore consider the very simple scattering process of Figure 2c, which is only a three-mode picture: two laser modes (1 and 2) coupled through one loss mode

(mode 3). As we will show now, this extremely simple system allows one to recover all features that lead to the Petermann excess noise factor.

Using now the mirrors transmission and reflection coefficients shown in Figure 2c, the three-mode unitary scattering matrix written in the orthogonal basis $\{1, 2, 3\}$ is:

$$\mathbf{M_3} = \begin{pmatrix} t_1 & 0 & r_1 \\ -r_1 r_2 & t_2 & t_1 r_2 \\ -r_1 t_2 & -r_2 & t_1 t_2 \end{pmatrix}. \quad (4)$$

The matrix truncated to modes 1 and 2 is then:

$$\mathbf{m_3} = \begin{pmatrix} t_1 & 0 \\ -r_1 r_2 & t_2 \end{pmatrix}. \quad (5)$$

It has eigenvalues t_1 and t_2, with respective eigenvectors:

$$\mathbf{v_1} = \begin{pmatrix} (t_2 - t_1)/(1 - t_1 t_2) \\ r_1 r_2/(1 - t_1 t_2) \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (6)$$

which are normalized but are clearly non orthogonal (let us remark that both components of $\mathbf{v_1}$ are smaller than 1 for any values of t_1, t_2). If the direction of propagation of the beam is reverted, the three-mode unitary matrix becomes:

$$\mathbf{L_3} = \begin{pmatrix} t_1 & -r_1 r_2 & -r_1 t_2 \\ 0 & t_2 & -r_2 \\ r_1 & t_1 r_2 & t_1 t_2 \end{pmatrix}. \quad (7)$$

Now, the matrix truncated to modes 1 and 2 becomes:

$$\mathbf{L_3} = \begin{pmatrix} t_1 & -r_1 r_2 \\ 0 & t_2 \end{pmatrix} \quad (8)$$

with the same eigenvalues t_1 and t_2 but different eigenvectors:

$$\mathbf{w_1} = \begin{pmatrix} (1 - t_1 t_2)/(t_2 - t_1) \\ 0 \end{pmatrix} \quad \mathbf{w_2} = \begin{pmatrix} r_1 r_2/(t_1 - t_2) \\ 1 \end{pmatrix}. \quad (9)$$

These eigenvectors are again non-orthogonal, but they are biorthogonal to the previous ones:

$$(\mathbf{v_1}, \mathbf{w_1}) = 1 \quad (\mathbf{v_2}, \mathbf{w_2}) = 1 \quad (10)$$

$$(\mathbf{v_1}, \mathbf{w_2}) = 0 \quad (\mathbf{v_2}, \mathbf{w_1}) = 0. \quad (11)$$

One recovers therefore all the mathematical ingredients which appear in the treatment of unstable resonators. We will comment on that from a physical point of view at the end of the paper.

The standard calculation of the excess noise factor then proceeds as follows: the eigenmode with the smallest losses (either t_1 or t_2) is taken as the lasing mode, while the other one is assumed to remain below threshold. The value of K_n for the lasing mode is given by equation (2), which yields:

$$K_1 = (\mathbf{w_1}, \mathbf{w_1}) = (1 - t_1 t_2)^2/(-t_1 + t_2)^2 \quad (12)$$

$$K_2 = (\mathbf{w_2}, \mathbf{w_2}) = 1 + (r_1 r_2)^2/(-t_1 + t_2)^2 = K_1. \quad (13)$$

This factor $K_1 = K_2 = K$ can become extremely large for small values of $(t_1 - t_2)$. Such a dramatic role of $(t_1 - t_2)$ may appear surprising for a passive system; however, one must remember that one mode is supposed to be lasing, which means that the laser gain g is clamped to the smallest of either $1/t_1$ or $1/t_2$, so that $g t_i = 1$ for the lasing mode. Then, small values of $(t_1 - t_2)$ just means that the non-lasing modes comes closer and closer to threshold, which explains the dramatic increase in the excess noise. This point will be discussed further in the following, from a quantum point of view.

An important issue in the discussions on the Petermann factor is that the lasing mode appears to have an excess noise which is much larger than its own vacuum noise. This is usually explained (see *e.g.* Ref. [1] and Refs. therein) by saying that this noise is not proper to the lasing mode, but is correlated with the noise in other modes. As it will be shown now, this argument is indeed quite true, but it still misses a point: this correlation has actually been induced by mode 3, which has been "hidden" up to now. By taking it into account explicitly, we will show now how to recover all previous results in a fully quantum picture.

3 The quantum approach

From the three-mode scattering matrix (Eq. (4)), one can deduce a quantum relationship for modes 1 and 2, but "truncating" the matrix is not the correct way to proceed in a quantum framework, because unitarity must be preserved. Moreover, in order to calculate an excess noise, we have to take explicitly into account the gain mechanism. We will therefore assume that modes 1 and 2 see the same gain g, so that the gain does not change the eigenvalues and eigenvectors of $\mathbf{m_3}$, while mode 3 sees no gain. Introducing the operators $\hat{a}_{\text{in}}$, $\hat{a}_{\text{out}}$ (mode 1), $\hat{b}_{\text{in}}$, $\hat{b}_{\text{out}}$ (mode 2), $\hat{c}_{\text{in}}$, $\hat{c}_{\text{out}}$ (mode 3) the input/output relations can be written:

$$\hat{a}_{\text{out}} = g t_1 \, \hat{a}_{\text{in}} + g r_1 \, \hat{c}_{\text{in}} + \sqrt{g^2 - 1} \, \hat{a}_{\text{s}}^\dagger \quad (14)$$

$$\hat{b}_{\text{out}} = g t_2 \, \hat{b}_{\text{in}} - g r_1 r_2 \, \hat{a}_{\text{in}} + g t_1 r_2 \, \hat{c}_{\text{in}} + \sqrt{g^2 - 1} \, \hat{b}_{\text{s}}^\dagger \quad (15)$$

where the spontaneous emission noise operators $\hat{a}_{\text{s}}^\dagger$, $\hat{b}_{\text{s}}^\dagger$ have been added to preserve unitarity in the amplifier [9].

In order to proceed further, one has also to introduce the other essential mechanism for the laser, which is cavity feedback. In order to keep the calculation as simple as possible, we will just assume here that each mode is recoupled onto itself using perfect mirrors, as shown in Figure 3a. A model including explicitly the output coupler, as shown in Figure 3b, is presented in Appendix A, and makes the calculations more intricate without changing any physical conclusion. From Figure 3 one can check easily the semiclassical results for the laser: if $t_2 > t_1$, mode 2 will be lasing first, and mode 1 will contain only amplified spontaneous emission noise. On the other hand, if $t_1 > t_2$, the laser will operate on a linear combination of the two cavity modes, given by $\mathbf{v_1}$ in equation (6).

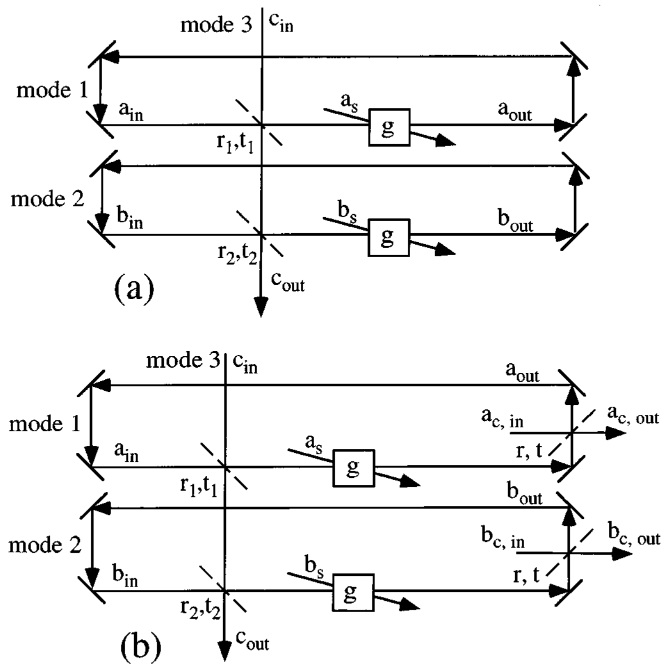

Fig. 3. (a): "Laser" built by gain and cavity feedback onto modes 1 and 2. Mode 3 remains a loss mode, so there is no laser output. (b): Same as (a), but the modes $\hat{a}_{c,out}$, $\hat{b}_{c,out}$ reflected off the (r, t) beamsplitters provide the laser output. The arrows on the gain boxes denote the "idle" spontaneous emission modes required by quantum mechanics.

As a first case, let us now assume that $t_2 > t_1$, and that gt_2 is very close to one, so that mode 2 is almost lasing. In order to keep the calculations simple, we use the following procedure: we assume that mode 1 is recoupled onto itself, as in Figure 3a, but that the cavity for mode 2 remains blocked. Then we calculate the spontaneous emission noise stored in mode 1 using the relation $\hat{a}_{out} = \hat{a}_{in}$, and we deduce the round-trip gain and noise for mode 2. This procedure makes the calculation much simpler, and will allow us to get the main results on the excess noise (see also the appendix). Using the relation $\hat{a}_{out} = \hat{a}_{in}$, one has:

$$\hat{a}_{in} = \frac{gr_1\ \hat{c}_{in} + \sqrt{g^2-1}\ \hat{a}_s^\dagger}{1-gt_1}. \tag{16}$$

Substituting into equation (15), one obtains:

$$\hat{b}_{out} = gt_2\ \hat{b}_{in} + gr_2\frac{t_1-g}{1-gt_1}\ \hat{c}_{in} + \sqrt{g^2-1}(\hat{b}_s^\dagger - \frac{gr_1r_2}{1-gt_1}\ \hat{a}_s^\dagger). \tag{17}$$

This equation, which describes the gain mechanism for mode b, has several very interesting features: first, it preserves the commutators from input to output, because it is easy to check that $[\hat{b}_{out}, \hat{b}_{out}^\dagger] = 1$ provided that all operators on the right hand side also verify the same commutation rule. Second, it shows that the spontaneous emission noise has considerably increased with respect to the value which can be expected from the gain. By comparing equations (15 and 17), the increase in the spontaneous emission noise power is:

$$K_b = 1 + \left(\frac{gr_1r_2}{1-gt_1}\right)^2. \tag{18}$$

When gt_2 is equal to one, it is obvious to check that K_b is just K. In that case, mode 2 is lasing when the cavity is unblocked, and describing the laser steady state would require to take into account the gain saturation mechanism, which is not our purpose here. We just mention that this can be done on the basis of equation (17), which is fully consistent quantum mechanically, and which does predict the expected amount of excess noise in the lasing mode.

Let us now consider the case where $t_1 > t_2$, with gt_1 very close to one. From the semiclassical calculations, the lasing mode is then the linear combination $\hat{d} = (t_2-t_1)/(1-t_1t_2)\ \hat{a} + r_1r_2/(1-t_1t_2)\ \hat{b}$. We introduce also the (non-lasing) mode $\hat{e} = r_1r_2/(1-t_1t_2)\ \hat{a} - (t_2 - t_1)/(1-t_1t_2)\ \hat{b}$ which is orthogonal to d. Equations (14 and 15) are then rewritten as:

$$\hat{e}_{out} = gt_2\ \hat{e}_{in} + gr_2\ \hat{c}_{in} + \sqrt{g^2-1}\ \hat{e}_s^\dagger \tag{19}$$

$$\hat{d}_{out} = gt_1\ \hat{d}_{in} - gr_1r_2\ \hat{e}_{in} + gt_2r_1\ \hat{c}_{in} + \sqrt{g^2-1}\ \hat{d}_s^\dagger. \tag{20}$$

It can be noticed that these equations have just the same form as equations (14 and 15), by changing a into e, b into d, and exchanging 1 and 2. Using the same calculations as above, the excess noise factor is $K_d = 1 + (gr_1r_2)^2/(-1 + gt_2)^2$, which is equal to K_1 given by equation (12), provided that $gt_1 = 1$.

An important point to be noticed is that $[\hat{b}_{out}, \hat{d}_{out}^\dagger] \neq 0$. Mathematically, this means that one cannot construct a quantum mechanically consistent set of modes which would include both $\hat{b}_{out}$ and $\hat{d}_{out}$: this is just the conceptual difficulty which was quoted at the beginning of the paper [6]. In more physical terms, the two situations considered above clearly require quite different operating conditions: $gt_2 = 1$ with $t_2 > t_1$ on one hand, and $gt_1 = 1$ with $t_1 > t_2$ on the other hand. Actually, these two situations are "incompatible" in a quantum mechanical sense, and modes $\hat{b}_{out}$ and $\hat{d}_{out}$ should not be considered as part of the same "set" of modes: depending on the respective values of t_1 and t_2, either one (but only one) is chosen as the lasing mode. Therefore, in a quantum framework, there is no "set" of non-orthogonal eigenmodes: depending on the parameter values the laser picks up one mode among several ones which are incompatible. Given this lasing mode, it is then possible to construct an orthogonal set which includes it: as shown above, it is $\{a, b, c, a_s, b_s\}$ for $t_2 > t_1$, and $\{d, e, c, d_s, e_s\}$ for $t_1 > t_2$.

It is also interesting to make a parallel between the semi-classical and quantum derivations of the excess noise factor. In order to follow closely the semi-classical derivation [5], one first writes equations (14 and 15) in a vector form:

$$\begin{pmatrix}\hat{a}_{\rm out}\\ \hat{b}_{\rm out}\end{pmatrix} = g\begin{pmatrix}t_1 & 0\\ -r_1r_2 & t_2\end{pmatrix}\begin{pmatrix}\hat{a}_{\rm in}\\ \hat{b}_{\rm in}\end{pmatrix} + g\begin{pmatrix}r_1\,\hat{c}_{\rm in}\\ t_1r_2\,\hat{c}_{\rm in}\end{pmatrix} + \sqrt{g^2-1}\begin{pmatrix}\hat{a}_{\rm s}^\dagger\\ \hat{b}_{\rm s}^\dagger\end{pmatrix}. \quad (21)$$

Looking for instance at the excess noise in eigenmode $\mathbf{v_2}$, the main step in the semi-classical calculation is to project this equation onto $\mathbf{w_2}$, which is the corresponding vector in the biorthogonal set [3,5]. Using equation (9), one gets:

$$\frac{r_1r_2}{t_1-t_2}\,\hat{a}_{\rm out} + \hat{b}_{\rm out} = gt_2\Big(\frac{r_1r_2}{t_1-t_2}\,\hat{a}_{\rm in} + \hat{b}_{\rm in}\Big) + gr_2\frac{1-t_1t_2}{t_1-t_2}\hat{c}_{\rm in} + \sqrt{g^2-1}\Big(\frac{r_1r_2}{t_1-t_2}\,\hat{a}_{\rm s}^\dagger + \hat{b}_{\rm s}^\dagger\Big). \quad (22)$$

This equation clarifies the role of the projection on the biorthogonal vector $\mathbf{w_2}$, which is at the heart of the semi-classical calculation: this projection actually picks up in other modes some extra terms, which are brought onto the mode amplitude b by the joint effect of losses and mode coupling. It results from this equation that the (normally-ordered) variance of the added noise is equal to $1+\big(\frac{r_1r_2}{t_1-t_2}\big)^2 = (\mathbf{w_2},\mathbf{w_2}) = K$, which is the essence of the semi-classical result. Equation (22) can also be given the same form as equation (17), by taking $gt_2 = 1$ and assuming $\hat{a}_{\rm in} = \hat{a}_{\rm out}$, as it was done previously. Under these conditions, both equations (17 and 22) can be written as:

$$\hat{b}_{\rm out} = \hat{b}_{\rm in} + \frac{r_2}{t_2}\left(\frac{1-t_1t_2}{t_1-t_2}\hat{c}_{\rm in} + \frac{r_1r_2}{t_1-t_2}\,\hat{a}_{\rm s}^\dagger + \hat{b}_{\rm s}^\dagger\right). \quad (23)$$

Further consequences of this equation, including the derivation of the laser linewidth, will be presented in Appendix A (case of two coupled cavities, each one with an outcoupling mirror) and Appendix B (case of one cavity with two outcoupling mirrors).

4 Discussion

Summarizing, two main new features actually emerge from this very simplified quantum description.

- First, the treatment of systems with non-orthogonal modes can be made fully consistent quantum mechanically by inserting appropriate loss and gain modes, in order to recover a unitary input-output matrix for a cavity roundtrip.

- Second, loss modes are not enough by themselves to get excess noise: what matters is the combined action of losses and mode coupling, which is translated into excess noise by the gain mechanism and the cavity feedback [12]. In order to make this notion more precise mathematically, let us consider a general unitary input-output matrix U including all modes, and two projection operators P and Q on the respective subsets of "laser" and "loss" modes (one has $P^2 = P$, $Q^2 = Q$, and $P + Q = 1$). It is easy to show that the restriction PUP of U to the laser modes has orthogonal eigenvectors if and only if :

$$PU^\dagger PUP = PUPU^\dagger P \quad (24)$$

which can also be written in the equivalent form:

$$PU^\dagger QUP = PUQU^\dagger P. \quad (25)$$

In order to get non-orthogonal laser eigenvectors, it is therefore necessary (but not sufficient) to have non-zero terms in the loss-dependant matrices $C = QUP$ or $D = QU^\dagger P$; the full condition for non-orthogonal laser eigenvectors is that $C^\dagger C \neq D^\dagger D$, which can be considered as the formal definition of loss-induced coupling. We note that this wording does not mean that the loss mechanism should also create coupling at the same time, but rather that laser modes are coupled (or correlated) by sharing common noise contributions due to the same loss modes.

An important point to be discussed is whether our toy model carries any relationship with transverse modes in laser resonators. It is clear that losses and mode coupling in a true resonator are not due to a beamsplitter, but rather to a combination of aperturing and diffraction effects. Nevertheless, we have shown above that non-orthogonal modes show up and relate to K in our toy model just in the same way as in true resonators: this creates a mathematical analogy between the two cases. From a physical point of view, one may try to explain why the excess noise factor has been observed in unstable resonators, and shows up to a much smaller extent if one introduces aperturing inside a stable cavity resonator [13,14]. From our results, this question can be rephrased as: what is the role of the cavity structure with respect to loss-induced coupling between cavity modes? One can then provide the following tentative answers:

- if there are no intracavity losses, there is no excess noise factor [12]; this situation is theoretically possible in stable cavities, but not in unstable ones, where aperturing is an essential ingredient;

- in a stable cavity, there may be losses but the mode coupling effects are usually very weak: the situation is then similar to the one depicted in Figure 2a. This is true for absorption losses, but also for aperturing losses, because the lasing mode (which has the smallest losses) will basically reshape to avoid the aperture. Therefore, the t_i for the lasing mode is very close to 1, r_i to zero, and K goes back to one (see Eq. 13 above). However, we point out that it was predicted [14] and observed [15] recently that significant excess noise may still appear in stable cavities, provided that diffraction losses are made very large. This is quite in agreement with our interpretation;

- in unstable cavities, the lasing mode cannot escape the aperturing, as it is obvious from the shape of the output beam from such lasers. Clearly, this is also associated with the fact that the semi-classical calculation shows large non-orthogonality between the modes. However, the physical picture which emerges from our "toy model" is that both the non-orthogonality of the cavity eigenmodes and the excess noise factor are actually consequences of the loss-induced mode coupling.

Finally, we point out that it should be possible relatively straightforwardly to include all transverse modes in the present model, in order to obtain a more realistic coupled-mode description of laser resonators. This method, which can be seen as a multi-transverse-mode extension of the calculations of references [10,11], could be useful to calculate for instance not only the linewidth [7] but also the quantum intensity and phase noises at the laser output, as well as the possible correlations between them. However, such a theory is beyond the scope of the present paper.

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Appendix A: Case of two modes with output coupling

We have considered above a very simplified three-modes pictures where the laser output coupler is essentially ignored. For the sake of completeness, it is interesting to extend slightly the model in order to include this output coupling; the usual input/output dynamics in the laser cavity could be included as well, as it is done in Appendix B. In order to do that, let us consider again the five modes picture of equation (3), but assuming that the reflection coefficient $r_4 = r_5 = r$ corresponds to the output coupler, which is supposed to have the same r and t for both modes 1 and 2. As shown in Figure 3b, we will denote as $\hat{a}_{\rm c,in}$, $\hat{b}_{\rm c,in}$ the input modes to the cavity (which are in the vacuum state) and $\hat{a}_{\rm c,out}$, $\hat{b}_{\rm c,out}$ the corresponding output modes. The following conditions have to be fulfilled:

$$\hat{a}_{\rm out} = gtt_1\ \hat{a}_{\rm in} + gtr_1\ \hat{c}_{\rm in} + t\sqrt{g^2-1}\ \hat{a}_{\rm s}^\dagger + r\hat{a}_{\rm c,in} \quad (A.1)$$

$$\hat{a}_{\rm c,out} = t\hat{a}_{\rm c,in} - r(gt_1\ \hat{a}_{\rm in} + gr_1\ \hat{c}_{\rm in} + \sqrt{g^2-1}\ \hat{a}_{\rm s}^\dagger) \quad (A.2)$$

$$\hat{b}_{\rm out} = gtt_2\ \hat{b}_{\rm in} - gtr_1r_2\ \hat{a}_{\rm in} + gtt_1r_2\ \hat{c}_{\rm in} + t\sqrt{g^2-1}\ \hat{b}_{\rm s}^\dagger + r\hat{b}_{\rm c,in} \quad (A.3)$$

$$\hat{b}_{\rm c,out} = t\hat{b}_{\rm c,in} - r(gt_2\ \hat{b}_{\rm in} - gr_1r_2\ \hat{a}_{\rm in} + gt_1r_2\ \hat{c}_{\rm in} + \sqrt{g^2-1}\ \hat{b}_{\rm s}^\dagger). \quad (A.4)$$

If $t_2 > t_1$, the (lasing) classical eigenmode remains mode 2; on the other hand, if $t_1 > t_2$, the lasing mode is the combination $\hat{d} = (t_2 - t_1)/(1 - t_1t_2)\ \hat{a} + r_1r_2/(1 - t_1t_2)\ \hat{b}$, where this equation can be used with any subscript for the operators. In the steady state, the cavity resonance conditions ensures that $\hat{a}_{\rm out} = \hat{a}_{\rm in}$, $\hat{b}_{\rm out} = \hat{b}_{\rm in}$, $\hat{d}_{\rm out} = \hat{d}_{\rm in}$. One can therefore eliminate the intracavity modes, and one obtains after a tedious but straightforward calculation:

$$\hat{b}_{\rm c,out} = \frac{(-t+gt_2)}{-1+gtt_2}\ \hat{b}_{\rm c,in} + \frac{gr^2r_1r_2}{(-1+gtt_1)(-1+gtt_2)}\ \hat{a}_{\rm c,in} + \frac{grr_2(gt-t_1)}{(-1+gtt_1)(-1+gtt_2)}\ \hat{c}_{\rm in} + \frac{r\sqrt{g^2-1}}{-1+gtt_2}\ \hat{b}_{\rm s}^\dagger + \frac{r\sqrt{g^2-1}\ gtr_1r_2}{(-1+gtt_1)(-1+gtt_2)}\ \hat{a}_{\rm s}^\dagger \quad (A.5)$$

$$\hat{d}_{\rm c,out} = \frac{(-t+gt_1)}{-1+gtt_1}\ \hat{d}_{\rm c,in} + \frac{gr^2r_1r_2}{(-1+gtt_1)(-1+gtt_2)}\ \hat{e}_{\rm c,in} + \frac{grr_1(gt-t_2)}{(-1+gtt_1)(-1+gtt_2)}\ \hat{c}_{\rm in} + \frac{r\sqrt{g^2-1}}{-1+gtt_1}\ \hat{d}_{\rm s}^\dagger + \frac{r\sqrt{g^2-1}\ gtr_1r_2}{(-1+gtt_1)(-1+gtt_2)}\ \hat{e}_{\rm s}^\dagger \quad (A.6)$$

where as previously the $\hat{e}$ operators correspond to the linear combination $\hat{e} = r_1r_2/(1-t_1t_2)\ \hat{a} - (t_2-t_1)/(1-t_1t_2)\ \hat{b}$. Let us remark that each of these equations conserves the input/output commutation rules, but that $[\hat{b}_{\rm c,out}, \hat{d}_{\rm c,out}^\dagger] \neq 0$. From equations (A.5 and A.6) the excess noise due to the mode coupling can be calculated to be:

$$K_b = 1 + \frac{(gtr_1r_2)^2}{(1-gtt_1)^2} \qquad K_d = 1 + \frac{(gtr_1r_2)^2}{(1-gtt_2)^2}. \quad (A.7)$$

One has to consider again two "incompatible" situations: if $t_2 > t_1$, mode b will be lasing, and $K_b = K$ for $gtt_2 = 1$; if $t_1 > t_2$, mode d will be lasing, and $K_d = K$ for $gtt_1 = 1$. We recover therefore all results of the main text in a more rigourous calculation, which treats the cavity feedback in a fully consistent way and looks at the input/output transfer as seen from outside the cavity. Finally, we note that even in the case of a laser with a single transverse mode mode, the observed linewidth may differ from the usual Schawlow-Townes value; this effect, which is known as the "longitudinal Petermann factor", is discussed in Appendix B.

Appendix B: Longitudinal Petermann factor

For completeness we consider in this section the case where the laser cavity has one single transverse mode, but two different input/output mirrors of arbitrary transmission coefficients. In that case the laser linewidth is in general different from the usual Schawlow-Townes formula, and the ratio between this formula and the correct value is known as the "longitudinal Petermann factor" [16–18]. The following calculation does not mean to be exhaustive (see *e.g.* [18] for a complete analysis), but rather to present a simple quantum approach of this effect, which turns out to be different from the previous transverse mode case.

In order to compare easily our results with previous works, we consider the situation depicted in Figure 4, where two mirrors $M_1(r_1, t_1)$ and $M_2(r_2, t_2)$ are separated by a medium with a gain g. We will assume the gain medium to be ideal, in the sense that the gain is constant

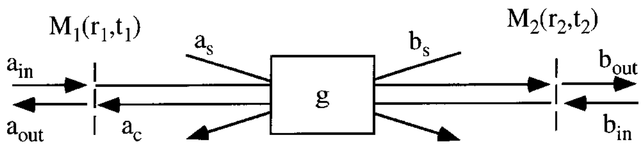

Fig. 4. Doubled-ended linear laser cavity with two mirrors $M_1(r_1, t_1)$ and $M_2(r_2, t_2)$. Mirror M_1 is supposed to be the normal output coupler, and M_2 may correspond to intracavity losses.

over the frequency range of interest, and that the added noise is equal to the minimum required by quantum mechanics for a laser amplifier (phase-insensitive amplifier according to the terminology of [9]). Under this hypothesis, the result does not depend on the precise structure of the gain medium (lumped or distributed), but only on the overall value of the laser gain, which is determined by the laser threshold condition $g^2 r_1 r_2 = 1$.

In order to obtain the expression of the laser linewidth, we express the variation of the field $\hat{a}_c$ inside the laser just before mirror M_1, during a round-trip time $\delta t = \tau_{rt} = 2L/c$ (L is the cavity length):

$$\tau_{rt} \frac{\delta \hat{a}_c}{\delta t} = (g^2 r_1 r_2\, \hat{a}_c + g^2 t_1 r_2\, \hat{a}_{in} + g t_2\, \hat{b}_{in} + g r_2 \sqrt{g^2-1}\, \hat{a}_s^\dagger + \sqrt{g^2-1}\, \hat{b}_s^\dagger) - \hat{a}_c \tag{B.1}$$

where a_s, b_s are uncorrelated spontaneous emission noises corresponding to the forward and backward ways through the amplifying medium. The noise of the phase quadrature inside the cavity can be characterized by the normally-ordered variance $:\Delta Y_c^2:$ of the out-of-phase noise $\delta Y_c = (\delta a_c - \delta a_c^\dagger)/(2i)$. Using the standard quantum phase-diffusion model [10,11], the laser linewidth is related to this quadrature phase noise by:

$$\Delta\omega_{QM} = \left(\frac{c}{2L}\right) \frac{:\Delta Y_c^2:}{2|\alpha_c|^2} \tag{B.2}$$

where α_c is the laser field (taken to be real) inside the cavity just before mirror M_1. This result can equivalently be expressed as a function of the output power on mirror M_1 in units of photons/s, which is $P_{1out} = t_1^2 |\alpha_c|^2/\tau_{rt}$:

$$\Delta\omega_{QM} = \left(\frac{c}{2L}\right)^2 \frac{t_1^2 :\Delta Y_c^2:}{2P_{1out}}. \tag{B.3}$$

Using equation B.1 with $g^2 r_1 r_2 = 1$, one obtains:

$$:\Delta Y_c^2: = \frac{1}{4}(g^2 r_2^2 (g^2-1) + (g^2-1)) = \frac{(r_1+r_2)(1-r_1 r_2)}{4 r_1^2 r_2} \tag{B.4}$$

and therefore the linewidth is:

$$\Delta\omega_{QM} = \left(\frac{c}{2L}\right)^2 \frac{t_1^2 (r_1+r_2)(1-r_1 r_2)}{8 r_1^2 r_2 P_{1out}} = \left(\frac{c}{2L}\right)^2 \frac{(r_1+r_2)^2 (1-r_1 r_2)^2}{8 (r_1 r_2)^2 P_{out}} \tag{B.5}$$

where we have introduced the total output power $P_{out} = P_{1out} + P_{2out}$ in units of photons/s, and used the relation $r_1 P_{1out}/t_1^2 = r_2 P_{2out}/t_2^2$ which is generally valid provided that $g^2 r_1 r_2 = 1$.

When $r_2 = 1$ (single-ended cavity), one has $P_{1out} = P_{out}$, and : $\Delta Y_c^2 := \frac{1}{4}(\frac{1}{r_1^2} - 1)$. Physically, this means that the amount of phase noise, and therefore the linewidth, have just the minimum value required for a phase-independant laser amplifier with a power gain $1/r_1^2$ [9]. We note that this remains true whatever is the structure of the gain medium inside the cavity. When $r_2 < 1$, the linewidth decreases for a given P_{out}, but *increases for a given* P_{1out}. This expresses the general fact that intracavity losses (here due to M_2) increase the output phase noise, as seen from the outcoupling mirror M_1.

Let us now compare these results to the Schawlow-Townes formula, written under the form:

$$\Delta\omega_{ST} = \frac{\Delta\omega_{cav}^2}{2P_{out}} = \left(\frac{c}{2L}\right)^2 \frac{\ln(r_1 r_2)^2}{2P_{out}} \tag{B.6}$$

where $\Delta\omega_{cav} = |\ln(r_1 r_2)|/\tau_{rt}$ is the cavity linewidth.

Using equation (B.5), the correction to equation (B.6) is therefore a multiplicative factor:

$$\frac{\Delta\omega_{QM}}{\Delta\omega_{ST}} = \left(\frac{(r_1+r_2)(1-r_1 r_2)}{2 r_1 r_2 \ln(r_1 r_2)}\right)^2 \tag{B.7}$$

which is a standard result [16,17]. As a conclusion, the quantum approach presented here includes automatically the "longitudinal" Petermann effect, and the phase noise and laser linewidth are directly related to the overall value of the laser gain. The present approach is therefore closer to the derivations presented in [16,18] (which use more general Green's functions methods) than in [17] (which uses non-orthogonal longitudinal eigenmodes).

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12. In principle, one could also have *gain-induced* coupling: this would require for instance that $\hat{a}_s \equiv \hat{b}_s$, *i.e.*, the spontaneous emission from one mode is recoupled into the other one. The important point is that the (loss or gain) mode which is used for coupling ($\hat{c}$ in our model, $\hat{a}_s \equiv \hat{b}_s$ for gain coupling) should not belong to the set of laser modes: otherwise, the change from uncoupled to coupled eigenmodes is unitary, and $K = 1$.
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